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1985 J. Phys. A: Math. Gen. 18 729

(http://iopscience.iop.org/0305-4470/18/4/020)

J. Phys. A: Math. Gen. 18 (1985) 729-733. Printed in Great Britain

COMMENT

Relativistic extensions of dynamical systems

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Received 17 October 1984

Abstract. Extensions of holonomic dynamical systems are presented as formal generalisations of special relativity. Their construction is based on a weak correspondence principle and a particular class of solutions is given explicitly. In some cases the symmetry content of the theory is explored and the existence of a conservation law is established. In addition, examples of dynamical systems which exhibit time reversibility and whose extensions are time-irreversible are briefly discussed.

Let M be an n-dimensional differentiable manifold which is connected and Hausdorff. Here M is identified with the configuration space of a holonomic dynamical system S_n with n degrees of freedom. If $(x^1 \ldots x^n)$ is a local coordinate system on the neighbourhood U of $x \in M$ the evolution of the system S_n, is described in this neighbourhood, by means of the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = F_i(x, \dot{x}), \qquad i = 1, \ldots, n \tag{1}$$

where $T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$, $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}t$ is the kinetic energy and F_i is the generalised force.

The *triplet* $D = (g_{ij}, F_i; t)$ describes the system, up to a choice of the initial conditions, in the neighbourhood of the point $x \in M$. Conversely it is assumed that any such triplet D, where g_{ij} is a tensor of rank n, locally defines a dynamical system S_n, in the neighbourhood U of the configuration space M, via (1). The triplet D will be called the (local) *description* of the system S_n in $U \subseteq M$. For a given description D the tensor g_{ij} introduces locally on M the geometry of a Riemannian space V_n and the equations of motion (1) can be written in the form

$$\frac{Du^i}{\mathrm{d}t} \equiv \frac{\mathrm{d}^2 x^i}{\mathrm{d}t} + \{{}_j{}^i{}_k\} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = F^i, \tag{1a}$$

where $\{{}_j{}^i{}_k\}$ are the Christoffel symbols of the second kind with respect to the metric g_{ij}.

The dynamical system S_n is by definition autonomous. Furthermore it is stipulated that the time parameter in the description D is a matter of convention. This means that another description $\tilde{D} = (\tilde{g}_{ij}, \tilde{F}; \tilde{t})$ may exist, leading to the same *paths* in U (i.e. the same trajectories when parametrisations are disregarded). Hence, D and $\tilde{D}$ describe dynamical systems which are *corresponding systems* in the sense of Painlevé, i.e. systems whose paths coincide in U (Painlevé 1894). Non-trivial examples of corresponding systems have been given by Levi-Civita (1896) and Agostinelli (1937). However, the important point for our purposes is that it can be proved (this proof will not be given here) that, under certain general requirements, the notion of corresponding systems naturally induces an *equivalence relation* on the set of all dynamical systems, which are defined on $U \subseteq M$. Clearly, this equivalence relation suggests a new point of view, where the conventional dynamical system is replaced (locally) by its equivalence class.

In this comment we shall introduce a specific approach which avoids the intricacies of the aforementioned 'equivalence problem' and which takes advantage of the arbitrariness in the choice of the time parameter. In particular we shall introduce an extra degree of freedom which will be identified with the time parameter in the local description D_U, i.e.

$$x^0 = ct \tag{2}$$

where c is an arbitrary constant. Equation (2) implies an extended neighbourhood $U \times R$, which can be regarded as the local trivialisation of the new configuration space M_1. This configuration space is not, in general, the product $M \times R$, where $R = (-\infty, +\infty)$ but rather a line-bundle over M. Now, a class of systems S_{n+1} can be constructed over the extended configuration space M_1, such that their paths and their time parameter are related to those of the original S_n. This construction will be presented into a step-by-step procedure and will be such that the paths of the two systems are related via the projection map $p: M_1 \to M$, where $p^{-1}(x) = R$ for each $x \in M$. The essential feature of this procedure is that the construction of the extensions S_{n+1} is based on a generalisation of the notion of equivalence introduced previously. Thus, a certain portion of the deformation class of the original system is finally realised by means of the extensions S_{n+1}.

Let $\check{D} = (\check{g}_{ab}, \check{F}_a; \tau)$, $a, b = 0, 1 \ldots n$, be the local description in $U \times R$ of a system S_{n+1}, defined over the extended configuration space M_1. Now the new time parameter is no longer arbitrary but it is defined in an invariant way, i.e.

$$\mathrm{d}s^2 = c^2\, \mathrm{d}\tau^2 \tag{3}$$

where $\mathrm{d}s^2$ is the element of length in the Riemannian space V_{n+1}, induced locally on M_1 via the tensor field $\check{g}_{ab}$. This means that the rate of change

$$\mathrm{d}t/\mathrm{d}\tau = f(t, x^i, \dot{x}^i), \tag{4}$$

can immediately be deduced from (3) in the coordinate systems

$$t' = t \qquad x'^k = x'^k(x^j). \tag{5}$$

Clearly f is *not* a scalar with respect to arbitrary coordinate transformations in V_{n+1}, but it is a scalar in V_n. Furthermore, equation (4) may be regarded as relating the coordinate time to the proper time τ along an arbitrary, but otherwise fixed, trajectory of the system S_{n+1}. If f is known the velocity of S_{n+1} can be expressed in terms of the velocity of S_n, i.e.

$$U^a = (cf, fu^k) \tag{6}$$

in the coordinate systems (5). Next it is assumed that in the coordinate systems (5) both f and the tensor $\check{g}_{ab}$ are expressed through g_{ik} and in terms of 'geometric objects' which are defined in V_n. This assumption restricts the possible forms of the extended tensor $\check{g}_{ab}$ in the coordinate systems (5), to the following cases:

$$(A_\pm): \qquad \check{g}_{ab} = \begin{pmatrix} \varphi^2 \pm \psi^2 & \psi_j \\ \psi_i & \pm g_{ij} \end{pmatrix}, \qquad \check{g}^{ab} = \begin{pmatrix} 1/\varphi^2 & \mp\psi^k/\varphi^2 \\ \mp\psi^j/\varphi^2 & \pm g^{jk} + (\psi^j\psi^k/\varphi^2) \end{pmatrix} \tag{7a}$$

$$(B_\pm):\qquad \check{g}_{ab}=\begin{pmatrix}0 & \psi_j\\ \psi_i & \pm\mu^2 g_{ij}\end{pmatrix},\qquad \check{g}^{ab}=\begin{pmatrix}\mp\mu^2/\psi^2 & \psi^k/\psi^2\\ \psi^j/\psi^2 & \pm\mu^{-2}[g^{jk}-(\psi^j\psi^k/\psi^2)]\end{pmatrix},\tag{7b}$$

$$(\Gamma_\pm):\qquad \check{g}_{ab}=\begin{pmatrix}\varphi^2 & 0\\ 0 & \pm\mu^2 g_{ij}\end{pmatrix},\qquad \check{g}^{ab}=\begin{pmatrix}\varphi^{-2} & 0\\ 0 & \pm\mu^{-2}g^{jk}\end{pmatrix},\tag{7c}$$

where $\psi^2=g^{ij}\psi_i\psi_j$. The ψ_j, φ and μ are arbitrary and in general they may depend on t. From (7) and (3) the form of f in the coordinate systems (5), is uniquely determined. In particular, for each case we have respectively

$$f_A=(\varphi^2\pm\psi^2+2c^{-1}\psi_j u^j\pm c^{-2}g_{ij}u^iu^j)^{-1/2}\tag{8a}$$

$$f_B=(2c^{-1}\psi_j u^j\pm c^{-2}\mu^2 g_{ij}u^iu^j)^{-1/2}\tag{8b}$$

$$f_\mu=(\varphi^2\pm c^{-2}\mu^2 g_{ij}u^iu^j)^{-1/2}\tag{8c}$$

where we have fixed the positive sign in the definition of f.

Equations (7) guarantee that in all cases the decomposition

$$\{{}_j{}^i{}_k\}_{\text{ex}}=\{{}_j{}^i{}_k\}+C^i_{jk}\tag{9}$$

holds, where the Christoffel symbols $\{\ \}_{\text{ex}}$ and $\{\ \}$ refer to the spaces V_{n+1} and V_n respectively and where C^i_{jk} is a tensor in V_n. Now the equations of motion of the system S_{n+1} can be written in the form

$$(\mathrm{D}u^i/\mathrm{d}t)+C^i_{jk}u^ju^k+2c\{{}_0{}^i{}_k\}u^k+c^2\{{}_0{}^i{}_0\}$$
$$=-(\mathrm{d}\ln f/\mathrm{d}t)u^i+f^{-2}\check{F}^i+f^{-1}\check{F}^i_ju^j+cf^{-1}\check{F}^i_0\tag{10a}$$

$$c(\mathrm{d}\ln f/\mathrm{d}t)+\{{}_j{}^0{}_k\}u^ju^k+2c\{{}_0{}^0{}_j\}u^j+c^2\{{}_0{}^0{}_0\}=f^{-2}\check{F}^0+f^{-1}\check{F}^0_ju^j+cf^{-1}\check{F}^0_0\tag{10b}$$

where the absolute derivative $\mathrm{D}/\mathrm{d}t$ refers to V_n (here the force $\check{F}^a$ has been assumed to be linear with respect to the velocities U^a). The decomposition (10) characterises the coordinate systems (5). In fact, in these systems the trajectories of S_{n+1} naturally split into a projection lying in the neighbourhood U and to a projection along the coordinate time axis. In addition, equations (10a) and (8) can be independently integrated and equation (10b) can be regarded simply as a compatibility condition.

The elimination of the term $-(\mathrm{d}\ln f/\mathrm{d}t)$ between (10a) and (10b) naturally introduces the *adjoint* system $\tilde{S}_n$ locally defined on M via (10a). The local description of the adjoint system in U is of the form $\tilde{D}=(g_{ij}, \tilde{F}_i; t)$. Thus, both systems refer to the same V_n and admit the same time convention. In addition, the adjoint system $\tilde{S}_n$ may be regarded as the image of the system S_{n+1} locally on M under the projection map p. Our requirement *that the systems S_n and $\tilde{S}_n$ are equivalent*, transfigures the projection map p into a *local homomorphism* of dynamical systems, in the sense that now p projects the paths of S_{n+1} on the paths of S_n. The above requirement is a kind of correspondence principle which relates the systems S_{n+1} and S_n. In fact this is the strongest correspondence principle which is not trivial. However, for our purposes it is convenient to introduce a weaker correspondence principle. Thus in the following we consider the notion of *pseudo-paths* of the system S_n, with respect to the pair (F^i, F^i_j). The equation of these curves is

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+\{{}_j{}^i{}_k\}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=\alpha F^i+\beta F^i_j\frac{\mathrm{d}x^j}{\mathrm{d}t}+\gamma\frac{\mathrm{d}x^i}{\mathrm{d}t}\tag{11}$$

where α, β and γ are functions of the velocities in general. The notion of pseudo-paths restricts the form of the involved force to that of the RHS of equation (11). Furthermore, equation (11) includes the paths of the system S_n up to equivalence and also generalises the notion of *subpaths* introduced by Yano (1944). In particular, equation (11) reduces to the equation of subpaths if $F^i_j = 0$. Now the weak correspondence principle requires that the *pseudo-paths of the systems* S_n *and* $\tilde{S}_n$ *are the same* (i.e. they refer to the same pair (F^i, F^i_j)). The class of systems which have the same pseudo-paths with S_n is the *deformation* class of S_n. Clearly, the deformation class of S_n includes its equivalence class as a proper subclass.

We shall call *relativistic extensions* of the system S_n the extensions S_{n+1} which result through the above procedure. An interesting class of relativistic extensions occurs in the special case

$$C^i_{jk} = 0, \qquad \check{F}^a = \check{F}^a_b = 0. \qquad (12a, b)$$

Comparing equations (10a) and (11) in the coordinate systems (5) and then identifying all terms which are not proportional to the velocity u^i, with the pair (F^i, F^i_j) for $\alpha = \beta = 1$, we get a class of *forceless* relativistic extensions S_{n+1}. In particular we have

$$(A): \quad \psi_{k,i} + \psi_{i,k} = 0, \qquad \partial_t \psi_j = \partial_t \varphi = 0 \qquad (13a, b)$$

$$F^i = -c^2\{{}_0{}^i{}_0\}_{\text{ex}} \qquad F^i_j = -2c\{{}_0{}^i{}_k\}_{\text{ex}} \qquad (14)$$

$$(B_\pm): \quad \psi_j(t, x) = \mu^2(t)\xi_j(x), \qquad \mu(t) = \exp(\lambda t) \qquad (15a, b)$$

$$\xi_{i,j} + \xi_{j,i} = \pm c^{-1}\lambda g_{ij} \qquad (16)$$

$$F^i = 0, \qquad F^i_j = \pm c^{-1}\{g^{il} - (\xi^i \xi^l/\xi^2)\}\{\xi_{l,j} - \xi_{j,l} \pm c^{-1}\lambda g_{lj}\} \qquad (17a, b)$$

$$(\Gamma): \quad (\alpha): \quad \mu = \mu(t), \qquad \varphi(t, x) = \mu(t)\sigma(x) \qquad (18a, b)$$

$$F^i = \pm\sigma g^{il}\sigma_{,l}, \qquad F^i_j = 0 \qquad (19a, b)$$

$$(\beta): \quad \mu = \mu(t), \qquad \varphi(t, x) = \varphi(t) \qquad (20a, b)$$

$$F^i = F^i_j = 0 \qquad (21)$$

$$(\gamma): \quad \mu(t) = \exp(-\lambda t/2), \qquad \varphi(t, x) = \varphi(t) \qquad (22a, b)$$

$$F^i = 0, \qquad F^i_j = \lambda\delta^i_j \qquad (\lambda = \text{constant}), \qquad (23a, b)$$

where the covariant differentiation ',' refers to the space V_n and where we have made use of the assumption that S_n is autonomous.

From (15) it is clear that the cases $\lambda = 0$ and $\lambda \neq 0$ correspond to conformal solutions. Thus, for the subsequent analysis it is sufficient to assume that $\lambda = 0$. Now from (13a) and (16) it is clear that relativistic extensions of type A and B are generated by the *Killing vectors* admitted by the space V_n. If in addition φ^2 is an invariant of the group of motions, the generator of the extension also generates a one-parameter subgroup of the group of motions, which leaves invariant the original system S_n, as well as its extension. Stated in a slightly different way this means that the Killing vector ξ_i in V_n naturally extends into a Killing vector ξ_a in V_{n+1}. However, extensions of type A and B have an additional symmetry, which can be identified as corresponding to a translation of the coordinate time. In all these cases there are conservation laws corresponding to a linear first integral of the equations of motion of the system S_{n+1}. In addition, in the case of extensions of type B the original system S_n also admits a linear first integral, as a result of the form of (17b)($\lambda = 0$).

Equations (7), (15*b*), (18*a*) and (20*a*) indicate that the extensions S_{n+1} do not exhibit *reversibility* with respect to the coordinate time, in general, although the original system may be *invariant under the time parameter reversal*: $t \to -t$. In particular, in the case of extensions of *A* and *B* types, the action of time-reversal changes the sign of the generator: $\xi_i \to -\xi_i$. These cases are rather exceptional. For example, a time-reversible system S_n admits extensions of type *A* if

$$\varphi^2 - \psi^2 = \text{constant}, \qquad \psi_j = \psi_{,j}, \qquad \psi_{,ij} = 0 \qquad (24a, b, c)$$

and extensions of type *B* if in addition to (24*b*) and (24*c*) $\lambda = 0$. Now condition (24*c*) means that the original V_n is a $V(0)$ or a $V(i)$ (Kruckovic 1961, 1967). On the other hand, in the cases where the extension is of type Γ, there are no restrictions about the original V_n; if the latter happens to be a space of constant curvature, then $\Gamma\beta$ extensions of the original system lead to a Robertson–Walker metric ($\varphi^2 = 1, n = 3$). In this particular case the extended system exhibits a time asymmetry defined through the sign of $\mathrm{d}\mu/\mathrm{d}t$.

In brief a prescription for the construction of relativistic extensions of holonomic systems has been given, based on a weak correspondence principle. In the cases where the extended metric is indefinite the resulting extensions introduce formal generalisations of special relativity. However, in all cases the extended systems can be regarded as forceless geometrisations of the original system. In particular, the extensions of $\Gamma_\pm$ type describe Newtonian systems in the limits where $u/c \ll 1$ and $V/c^2 \ll 1$. Here $\varphi^2 = 1 \mp 2V/c^2$, $\mu^2 = 1$ and the deformation induced by the construction now becomes negligible ($f \simeq 1$). In some cases the construction leads to extended systems that exhibit a time asymmetry, although the original one was invariant under time reversal. These examples are non-trivial since friction terms have *not* been included in the formalism of the original system. On the contrary, they indicate that the time-asymmetry of the extension is a manifestation of the fact that the deformation class of the original system includes non-reversible systems. This point of view should be compared with an algebraic approach to the problem of irreversibility recently introduced (Constantopoulos and Ktorides 1984). In fact the details of algebraically deformed members, within the deformation class of the original system, have been deliberately avoided here. However, their influence has been preserved, more or less, within the structure of the extension.

It is a pleasure to thank Dr C N Ktorides for valuable discussions and interesting remarks.

References

Agostinelli C 1937 *Memorie del R. Instituto Lombardo Di Scienze e Lettere, vol XXIII–XIV, serie III, fasc V*
Constantopoulos J P and Ktorides C N 1984 *J. Phys. A: Math. Gen.* **17** L29
Kruckovic G I 1961 *Sibirsk Mat. Z.* **2** 400 (in Russian)
—— 1967 *Trudy Vsesojuz. Zaocn Energet Inst. Vyp* **33** 3 (in Russian)
Levi-Crivita T 1896 *Annals di Matematica*, (ser. 2) **24** 255
Painlevé 1894 *Journal de Liouville* vol X
Yano K 1944 *Proc. Imp. Acad. Tokyo* **20** 340–5